APPROXIMATION OF FRACTIONAL INTEGRALS BASED ON B-SPLINE INTERPOLATION

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1 Introduction

In this paper, we propose a new approach to the numerical evaluation of the fractional integral operators. The presented methodology is performed by utilizing the well-known B-spline interpolation [2].

2 Fractional integrals

In this section, we introduce the fractional operators used in this work. According to the fractional calculus [1, 3] we recall the definitions of the left and right Riemann-Liouville fractional integrals for $\alpha > 0$

$$I_{a^+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(\tau)}{(x-\tau)^{1-\alpha}} d\tau \quad (x > a) \tag{1}$$

$$I_{b^-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(\tau)}{(\tau-x)^{1-\alpha}} d\tau \quad (x < b) \tag{2}$$

where Γ denotes the Gamma function.

3 Main results

The interval $[a,b]$ is divided into N sub-intervals $[x_i, x_{i+1}]$, for $i = 0,1,.,N-1$ with a constant step $h = (b-a)/N$ by using nodes $x_i = a + ih$. Next, we replace function f by the following expression

$$f(x) \approx \sum_{j=-1}^{N+1} K_j B_j(x) \tag{3}$$

where the B-splines are defined in the following way

$$B_j(x) = \frac{1}{h^3} \begin{cases} (x-x_{j-2})^3 & x_{j-2} \le x < x_{j-1} \\ h^3 + 3h^2(x-x_{j-1}) + 3h(x-x_{j-1})^2 - 3(x-x_{j-1})^3 & x_{j-1} \le x < x_j \\ h^3 + 3h^2(x_{j+1}-x) + 3h(x_{j+1}-x)^2 - 3(x_{j+1}-x)^3 & x_j \le x < x_{j+1} \\ (x_{j+2}-x)^3 & x_{j+1} \le x < x_{j+2} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

and coefficients $K_{-1}, K_0, \ldots, K_{N+1}$ are obtained by solving the matrix equation

$$\begin{bmatrix} -\frac{3}{h} & 0 & \frac{3}{h} & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & & \\ 0 & \cdots & 0 & 1 & 4 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 4 & 1 \\ 0 & \cdots & 0 & 0 & \frac{3}{h} & 0 & -\frac{3}{h} \end{bmatrix} \begin{bmatrix} K_{-1} \\ K_0 \\ \vdots \\ K_N \\ K_{N+1} \end{bmatrix} = \begin{bmatrix} f'(x_0) \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \\ f'(x_N) \end{bmatrix} \tag{5}$$

We put the interpolation (3) into expressions (1)-(2) and by the additivity of integration we get the approximations of analysed fractional operators. For the left Riemann-Liouville integral we have

$$\left. I_{a^+}^{\alpha} f(x) \right|_{x=x_i} \approx \sum_{j=-1}^{N+1} K_j I_{a^+}^{\alpha} B_j(x) \tag{6}$$

and for the right Riemann-Liouville integral we obtain

$$\left. I_{b^-}^{\alpha} f(x) \right|_{x=x_i} \approx \sum_{j=-1}^{N+1} K_j I_{b^-}^{\alpha} B_j(x) \tag{7}$$

The numerical results obtained for operator $I_{1^-}^{\alpha} f(x)$ and different values of order α are presented below

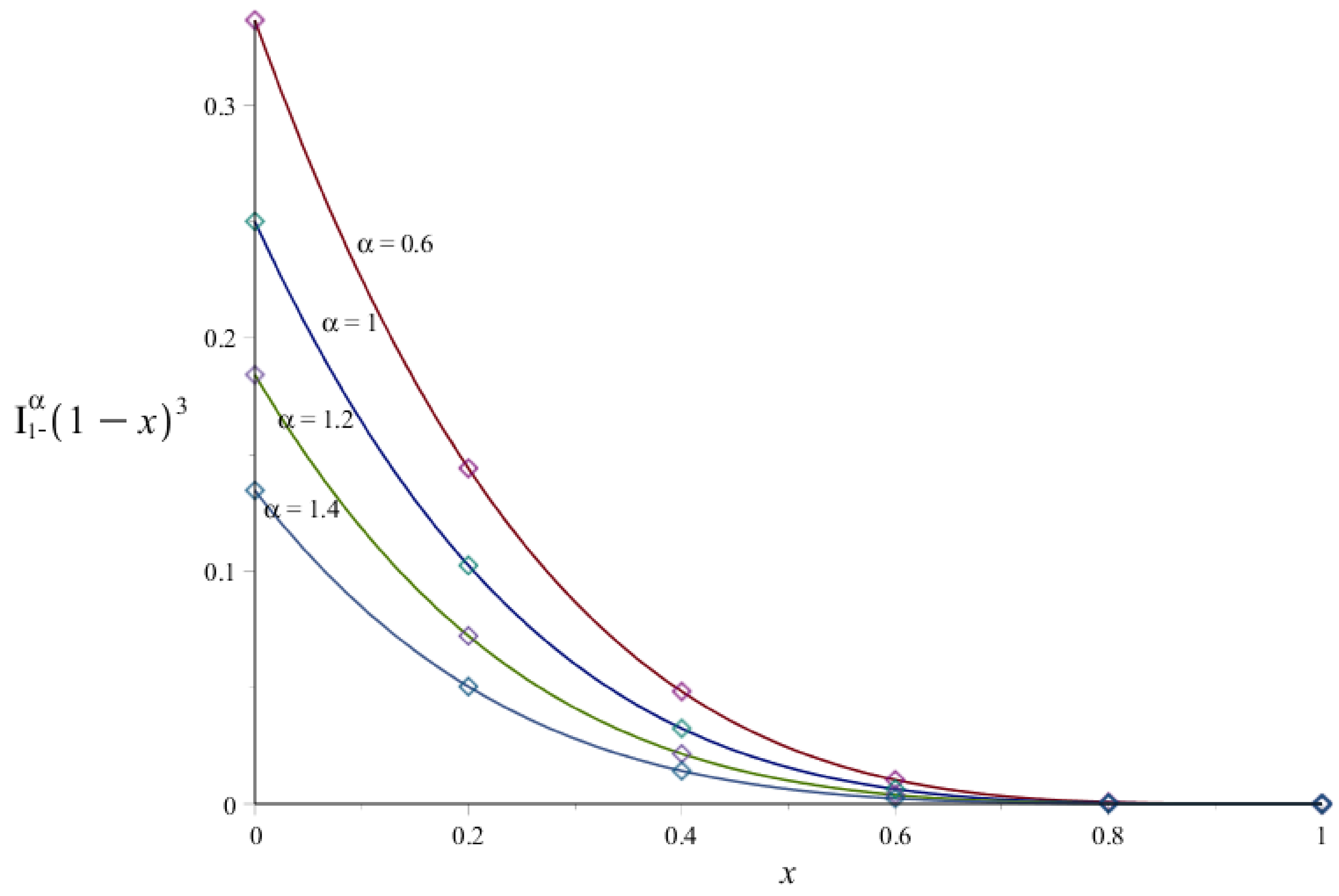

Figure 1: Numerical (points) and analytical (lines) results for different values of α.

Table 1: Errors and rates of convergence p generated by presented method for the integral $I_{1^-}^{\alpha}(1-x)^5$ at points $t \in \{0, 0.5, 0.75\}$ for $\alpha \in \{0.6, 0.8, 1, 1.2\}$

		$x=0$		$x=0.5$		$x=0.75$	
α	$h=1/N$	err	p	err	p	err	p
0.6	1/4	4.10E-04	-	1.21E-04	-	1.17E-04	-
	1/8	2.64E-05	3.95	7.89E-08	3.94	2.59E-06	5.50
	1/16	1.69E-06	3.97	5.34E-07	3.89	1.63E-07	3.99
	1/32	1.08E-07	3.98	3.46E-08	3.95	1.10E-08	3.89
	1/64	6.80E-09	3.98	2.21E-09	3.97	7.14E-10	3.95
0.8	1/4	3.78E-04	-	1.13E-04	-	6.94E-05	-
	1/8	2.38E-05	3.99	6.42E-06	4.14	2.09E-06	5.05
	1/16	1.50E-06	3.99	4.21E-07	3.93	1.15E-07	4.18
	1/32	9.42E-08	3.99	2.68E-08	3.97	7.56E-09	3.93
	1/64	5.90E-09	4.00	1.69E-09	3.99	4.81E-10	3.97
1	1/4	3.25E-04	-	1.01E-04	-	3.97E-05	-
	1/8	2.03E-05	4.00	4.88E-06	4.37	1.62E-06	4.62
	1/16	1.27E-06	4.00	3.15E-07	3.95	7.62E-08	4.41
	1/32	7.95E-08	4.00	1.98E-08	3.99	4.92E-09	3.95
	1/64	4.97E-09	4.00	1.24E-09	4.00	3.10E-10	3.99
1.2	1/4	2.69E-04	-	8.60E-05	-	2.20E-05	-
	1/8	1.68E-05	4.00	3.55E-06	4.60	1.20E-06	4.20
	1/16	1.05E-06	4.00	2.27E-07	3.96	4.83E-08	4.63
	1/32	6.56E-08	4.00	1.43E-08	3.99	3.09E-09	3.96
	1/64	4.10E-09	4.00	8.92E-10	4.00	1.94E-10	3.99

4 Conclusions

In this paper new formulas for numerical calculation of fractional integrals were presented. We derived the numerical schemes for the left and the right Riemann-Liouville fractional integrals utilizing B-spline interpolation. Finally, examples of numerical evaluations of analyzed operators, errors generated by the presented method and rate of convergence for the selected function are shown.

References

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Ministerstwo Nauki i Szkolnictwa Wyższego

Organizacja IX Konferencji Modelowanie Matematyczne w Fizyce i Technice (MMFT 2017)
- zadanie finansowane w ramach umowy 829/P-DUN/2017
ze środków Ministra Nauki i Szkolnictwa Wyższego przeznaczonych
na działalność upowszechniającą naukę.